Linear partial differential equations of first order

Definitions 9. A differential equation involving partial derivatives p and q only is of order one. (but it may involve powers of p and q).

And if the power of p and q is also one, then equation is known as **Linear partial differential equation of order one.**

Definitions 10. Lagrange's Linear Equation. A partial differential equation of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z (which is or first order and linear in p and q) is known as **Lagrange's Linear Equation.**

e.g., $(y+z)p + (z+x)q = x+y$ is a Lagrange's Linear equation

Art-6. To solve Lagrange's Linear Equation

Let $Pp + Qq = R$ ...(*i*)

be a Lagrange's linear equation where P, Q, R are functions of x, y, z

Now the system of equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ...(*ii*)

is called **Lagrange's system of ordinary differential equations for (*i*)**

Let $u = a$ and v and b two independent equations.

We have already shown (on Page 28) that relation

$f(u, v) = 0$ ...(*iii*)

Give rise to *a* partial differential equation $Pp + Qq = R$ ...(*iv*) where $P = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial z} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \frac{\partial(u,v)}{\partial(y,z)}$

and $Q = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial v}{\partial z}\cdot\frac{\partial u}{\partial x} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial(u,v)}{\partial(z,x)}$

and $R = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$

Thus $f(u, v) = 0$ is general solution of (*iv*)

Now find *u* and *v* to have the required solution

Differentiating $u = a$ and $v = b$, we get

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \text{ and } \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$$

Solving these, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial y}\frac{\partial u}{\partial z}} = \frac{dy}{\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial z} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

(by cross Multiplication method)

$$\Rightarrow \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(v)$$

which is equation (*ii*)

Thus solution of (*ii*) is given by $u = a$ and $v = b$. Hence if $u = a$ and $v = b$ are two independent solutions of

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, Then $f(u, v) = 0$ is general sol. of Lagrange's Linear Equation.

Note : Equation (*ii*) *i.e.,* $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

are **auxiliary equations or subsidiary equations**

Working Method to solve $Pp + Qq = R$ ...(*i*)

Step (A) : Write given equation in the form of (*i*) and find P, Q, R.

Step (B) : Write Lagrange's auxiliary equations

Step (C) : Find two independent solutions $u = a$ and $v = b$ of eqs. given in step (B)

Step (D) : Write $f(u, v) = 0$, which is general solution (integral) of the given equation.

ILLUSTRATIVE EXAMPLES

TYPE–I

Note : In this example, solution of $Pp + Qq = R$ is obtained by taking two members of auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ at a time and then integrating to have two independent solutions in variables whose differentials are involved in equation.

Example 1. Solve the following for general solution

(i) $p + q = \cos x$ (ii) $\alpha p + \beta q = \gamma$

(iii) $6p + 7q = 8$ (iv) $pz = x$

(v) $px + qy = 5z$

(vi) $x^2 p + y^2 q = z^2$

(vi) $yzp + zxq = xy$

(viii) $y^2 p + x^2 q = x^2 y^2 z^2$

(ix) $x^2 p + y^2 q = x^2 y^2 z^2$ (x) $y^2 zp + zx^2 q = xy^2$

(xi) $(x - \alpha) p + (y - \beta) q = z - \gamma$ (xii) $y^2 p - xyq = x(z - 2y)$

(xiii) $p \tan x + q \tan y = \tan z$ (xiv) $p + q = \frac{z}{\alpha}$

(xv) $pz = y$ (xvi) $p + q = \sin x$

Sol. (i) We are given $p + q = \cos x$

Compare it with $Pp + Qq = R$

Here $P = 1, Q = 1, R = \cos x$

$\therefore$ Auxiliary (subsidiary) equation are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\cos x} \quad \text{...(i)}$$

Taking first two members of (i), we get

$$dx = dy$$

Integrating $x = y + a$ or $\boldsymbol{x - y = a}$ ...(ii)

Now taking first and last two members of (i), we get

$$\cos x \, dx = dz$$

Integrating $\sin x = z + b$ or $\sin x - z = b$ ...(iii)

Thus from (ii) and (iii), we got $u = a$ and $v = b$

where $u(x, y, z) = x - y$, $v(x, y, z) = \sin x - z$

$\therefore$ **General sol of given equation is**

$f(x - y, \sin x - z) = 0$ where f is any arbitrary function.

or $\sin x - z = f(x - y)$

(*ii*) We are given $\alpha p+\beta q=\gamma$

Compare it with $P p+Q q=R$

Here $P=\alpha$, $Q=\beta$ and $R=\gamma$

$\therefore$ Auxiliary equations are

$$\frac{dx}{\alpha}=\frac{dy}{\beta}=\frac{dz}{\gamma} \quad \ldots(i)$$

Taking first two members of (*i*), we get

$$\beta\, dx=\alpha\, dy$$

Integrating $\beta x=\alpha y+a$ or $\beta x-\alpha y=a$ ...(*ii*)

Taking last two members of (*i*), we get

$$\gamma\, dy=\beta\, dz$$

Integrating $\gamma x=\beta z+b$ or $\gamma y-\beta z=b$ ...(*iii*)

Thus from (*ii*) and (*iii*), we get $u=a$ and $v=b$

where $u(x, y, z)=\beta x-\alpha y$, $v(x, y, z)=\gamma y-\beta z$

$\therefore$ **General sol.** of given equation is

$$f(\beta x-\alpha y, \gamma y-\beta z)=0, \quad \ldots(iv)$$

where f is any arbitrary function.

Note. Here if we take first and last members of (*i*)

we get $\frac{dx}{\alpha}=\frac{dz}{\gamma}$

$\Rightarrow$ $\gamma\, dx=\alpha\, dz$ integrating

$$\gamma x=\alpha z+b \quad \text{or} \quad \gamma x-\alpha z=b \quad \ldots(v)$$

$\therefore$ from (*ii*) and (*v*)

General sol is

$$f(\beta x-\alpha y, \gamma x-\alpha z)=0 \quad \ldots(vi)$$

Both (*iv*) and (*vi*) are general solutions of given equation since they give rise to same partial differential equation which is given

(*iii*) We are given $6p+7q=8$

compare it with $P p+Q q=R$

Here $P=6$, $Q=7$, $R=8$

$\therefore$ Auxiliary equations are $\frac{dx}{6}=\frac{dy}{7}=\frac{dz}{8}$ ...(*i*)

Taking first two members of (*i*), we get

$$7\, dx=6\, dy$$

Integrating $7x=6y+a$ or $7x-6y=a$ ...(*ii*)

Taking first and third members of (*i*)

we get $8\, dx=6\, dz$

$$\Rightarrow \quad \frac{1}{x} - \frac{1}{y} = a \quad \ldots(ii)$$

Taking last two members of (i) we get

$$\frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{or} \quad y^{-2}\,dy = z^{-2}\,dz$$

Integrating, we get $\frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} - b$

$$\Rightarrow \quad \frac{1}{y} - \frac{1}{z} = b \quad \ldots(iii)$$

From (ii) and (iii) we get $u = a$ and $v = b$

where $u(x, y, z) = \frac{1}{x} - \frac{1}{y}$, $v(x, y, z) = \frac{1}{y} - \frac{1}{z}$

$\therefore$ The general sol is given by

$f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$, where f is any arbitrary function

(vii) We are given

$$yzp + zxq = xy$$

Compare it with $\mathrm{P}\,p + \mathrm{Q}\,q = \mathrm{R}$

where $\mathrm{P} = yz$, $\mathrm{Q} = zx$, $\mathrm{R} = xy$

The auxiliary equation are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy} \quad \ldots(i)$$

Taking first and second members of (i)

We get $\frac{dx}{yz} = \frac{dy}{zx}$ or $x\,dx = y\,dy$

Integrating, we get

$$\frac{x^2}{2} = \frac{y^2}{2} + \frac{a}{2} \quad \text{or} \quad x^2 - y^2 = a \quad \ldots(ii)$$

Taking first and last members of (i)

we get $\frac{dx}{yz} = \frac{dz}{xy}$ or $x\,dx = z\,dz$

Integrating, we get $\frac{x^2}{2} = \frac{z^2}{2} + \frac{b}{2}$

$$\text{or} \quad x^2 - z^2 = b \quad \ldots(iii)$$

From (ii) and (iii) we get $u = a$, $v = b$

where $u(x, y, z) = x^2 - y^2$ and $v(x, y z) = x^2 - z^2$

$\therefore$ The general sol is given by

$f(x^2 - y^2, x^2 - z^2) = 0$ where f is any arbitrary function

or $$x^2 - z^2 = f(x^2 - y^2)$$

(*viii*) We are given

$$y^2 p + x^2 q = x^2 y^2 z^2$$

Compare it with $P p + Q q = R$

Here $P = y^2$, $Q = x^2$ and $R = x^2 y^2 z^2$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2 y^2 z^2} \quad ...(i)$$

Taking first and second members a of (*i*)

we get $x^2\, dx = y^2\, dy$

Integrating $\dfrac{x^3}{3} = \dfrac{y^3}{3} + \dfrac{a}{3} \Rightarrow x^3 - y^3 = a$ ...(*ii*)

From first and last members of (*i*)

we get, $$\frac{dx}{y^2} = \frac{dz}{x^2 y^2 z^2} \quad \text{or} \quad x^2\, dx = z^{-2}\, dz$$

Integrating $\dfrac{x^3}{3} = \dfrac{z^{-1}}{-1} + \dfrac{b}{3}$ or $x^3 + 3z^{-1} = b$ ...(*iii*)

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u(x, y, z) = x^3 - y^3$, $v(x, y, z) = x^3 + 3z^{-1}$

$\therefore$ The general sol of (*i*) is given by

$$f(x^3 - y^3, x^3 + 3z^{-1}) = 0 \quad \text{or} \quad x^3 + 3z^{-1} = f(x^3 - y^3).$$

(*ix*) We are given $x^2 p + y^2 q = \dfrac{x^2}{z^2}$.

Compare it with $P p + Q q = R$

Here $P = x^2$, $Q = y^2$ and $R = \dfrac{x^2}{z^2}$

The auxiliary Equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(i)$$

or $$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{\frac{x^2}{z^2}}$$

From first and second members of (*i*)

we get $x^{-2}\,dx = y^2\,dy$

Integrating, we get

$$\frac{x^{-2+1}}{-2+1} = \frac{y^{-2+1}}{-2+1} + a \quad \text{or} \quad -\frac{1}{x} + \frac{1}{y} = a$$

From first and last members of (*i*)

we get $\dfrac{dx}{x^2} = \dfrac{dz}{\dfrac{x^2}{z^2}}$ or $dx = z^{-2}\,dz$

Integrating $x = \dfrac{z^{-2+1}}{-2+1} + b$ or $\dfrac{1}{z} + x = b$

∴ The general sol is $f\left(\dfrac{1}{y} - \dfrac{1}{x}, x + \dfrac{1}{z}\right) = 0.$

(x) We are given $y^2 z\,p + z\,x^2\,q = x\,y^2$

Compose it with $p\,P + q\,Q = R$

Here $P = y^2 z$, $Q = z\,x^2$ and $R = x\,y^2$

The auxiliary Eqs. are

$$\frac{dx}{y^2 z} = \frac{dy}{z\,x^2} = \frac{dz}{x\,y^2}$$ ...

Taking first two members of (*i*)

We get $x^2\,dx = y^2\,dy$

Integrating $\dfrac{x^3}{3} = \dfrac{y^3}{3} + \dfrac{a}{3} \Rightarrow x^3 - y^3 = a$ ...(

Further taking first and last members of (*i*)

we get $x\,dx = z\,dz$

Integrating $\dfrac{x^2}{2} = \dfrac{z^2}{2} + \dfrac{b}{2} \Rightarrow x^2 - z^2 = b$ ...(iii

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u\,(x, y, z) = x^3 - y^3$ and $v\,(x, y, z) = x^2 - z^2$

∴ The general sol is given by

$$f(x^3 - y^3, x^2 - z^2) = 0$$

where f is any arbitrary function

(xi) We are given

$$(x - \alpha)\,p + (y - \beta)\,q = z - \gamma$$

Compare it with $P\,p + Q\,q = R$

Here $P = x - \alpha$, $Q = y - \beta$ and $R = z - \gamma$

$\Rightarrow \quad \dfrac{dy}{-y} = \dfrac{dz}{z-2y} \quad \Rightarrow (z-2y)\, dy = -y\, dz$

$\Rightarrow z\, dy + y\, dz = 2y\, dy \Rightarrow d\,(yz) = d\,(y^2)$

Integrating $yz = y^2 + b$

$\Rightarrow \quad yz - y^2 = b$ (ii)

From (*i*) and (*ii*), we get $u = a$ and $v = b$

where $u\,(x, y, z) = x^2 + y^2$ and $v\,(x, y, z) = yz - y^2$

$\therefore$ The general sol. is given by

$f(x^2 + y^2, yz - y^2) = 0.$

(***xiii***) We are given $p \tan x + q \tan y = \tan z$

Compare it with $Pp + Qq = R$

Here $P = \tan x$, $Q = \tan y$, $R = \tan z$

$\therefore$ The auxiliary eqs. are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$

Taking first two members of (*i*), we get

$\cot x\, dx = \cot y\, dy$

Integrating $\log \sin x = \log \sin y + \log a$

$\Rightarrow \quad \log \sin x - \log \sin y = \log a$

$\Rightarrow \quad \log \dfrac{\sin x}{\sin y} = \log a \Rightarrow \dfrac{\sin x}{\sin y} = a$ ...(ii)

Taking last two members of (*i*), we get

$\cot y\, dy = \cot z\, dz$

Integrating $\log \sin y = \log \sin z + \log b$

$\Rightarrow \quad \log \sin y - \log z = \log b$

$\Rightarrow \quad \log \dfrac{\sin y}{\sin z} = \log b \Rightarrow \dfrac{\sin y}{\sin z} = b$ ...(iii)

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u\,(x, y, z) = \dfrac{\sin x}{\sin y}$ and $v\,(x, y, z) = \dfrac{\sin y}{\sin z}$

$\therefore$ The general sol is given by

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

(***xiv***) We are given $\quad p + q = \dfrac{z}{\alpha}$

Compare it with $Pp + Qq = R$

Here $P = 1$, $Q = 1$ and $R = \dfrac{z}{\alpha}$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\frac{z}{\alpha}} \qquad \ldots(i)$$

Taking first two members of (i)

we get $dx = dy$

Integrating $x = y + a$ or $x - y = a$ $\qquad \ldots(ii)$

Taking last two members of (i)

we get $\frac{1}{\alpha} dy = \frac{dz}{z}$

Integrating $\frac{1}{\alpha} y = \log z + \log b$

$\Rightarrow \quad \frac{y}{\alpha} = \log (z\,b) \quad \Rightarrow \quad e^{y/\alpha} = z\,b$

$\Rightarrow \quad \frac{1}{z} e^{y/\alpha} = b$ $\qquad \ldots(iii)$

From (ii) and (iii), we get $v = a$ and $v = b$

where $u(x, y, z) = x - y$ and $v(x, y, z) = \frac{1}{z} e^{y/\alpha}$

$\therefore$ The general sol is.

$f\left(x - y, \frac{1}{z} e^{y/\alpha}\right) = 0$ where f is any arbitrary function

or $\quad \frac{1}{z} e^{y/\alpha} = f(x - y) \quad$ or $\quad e^{y/\alpha} = z f(x - y)$.

(xv) We are given $p\,z = y$

Compare it with $p\,P + q\,Q = R$

Here $P = z$, $Q = 0$, $R = y$

The auxiliary equations are $\qquad \ldots(i)$

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{y} \qquad \ldots(ii)$$

From first two members of (i), $dy = 0$ integrating $y = a$

From Ist and last member of (i) $\qquad \ldots(iii)$

we have $y\,dx = z\,dz \Rightarrow 2\,y\,dx = 2\,z\,dz$

Integrating $2\,y\,x = z^2 + b \Rightarrow 2\,y\,x - z^2 = b$

From (ii) and (iii), we get $u = a$ and $v = b$

where $u(x, y, z) = y$ and $v(x, y, z) = 2\,y\,x - z^2$

$\therefore$ The general sol is given by

$$f(y, 2yx - z^2) = 0$$

$\therefore$ where f is any arbitrary function

or $2yx - z^2 = f(y)$ any arbitrary function

(*xvi*) We are given $p + q = \sin x$

Compare it with $Pp + Qq = R$

Here $P = 1,\ Q = 1,\ R = \sin x$

$\therefore$ Auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sin x} \quad ...(i)$$

Taking first two members of (*I*), we get

$$dx = dy$$

Integrating $x = y + a$ or $x - y = a$ ...(*ii*)

Now taking first and last members of (*i*), we get

$$\text{Sin } x\, dx = dz$$

Integrating $-\cos x + b = z$ *or* $z + \cos x = b$ ....(*iii*)

Thus from (*ii*) and (*iii*), we get $u = a,\ v = b$

Where $u = x - y,\ v = z + \cos x$

$\therefore$ General sol of given equation is

$F(x - y, z + \cos x) = 0$ or $z + \cos x = f(x - y)$ where f is any arbtrary function

TYPE –II

In the next example, solution of $Pp + Qq = R$ is obtained by taking two members of the auxiliary equation and integrate to have an equation (one independent solution) in the variables whose differentials are involved and another independent solution is obtained by making use of the first solution (integral).

Example 2. Solve the following Lagrange's linear equations for general solution.

(*i*) $p - q = \log(x + y)$

(*ii*) $(p - q)(x + y) = z$

(*iii*) $xzp + yzq = xy$

(*iv*) $zp - zq = x + y$

(*v*) $xy^2 p - y^3 q = -\alpha xz$

(*vi*) $p - 2q = 3x^2 \sin(2x + y)$

(*vii*) $z(p - q) = z^2 + (x + y)^2$

(*viii*) $p + 3q = 5z - \tan(3x - y)$

(*ix*) $p + 5q = z - \cot(5x - y)$

(*x*) $5p - 6q = 5x^4 \cos(6x + 5y)$

(*xi*) $px + qz = -y$

Sol. (*i*) We are given $p - q = \log(x + y)$ diff. equation

Compare it with $Pp + Qq = R$

Here $P = 1,\ Q = -1,\ R = \log(x + y)$

$\therefore$ The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or $$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad ...(i)$$

From first two members of (*i*)

We get $dx + dy = 0$

Integrating $x + y = a$ ...(*ii*)

From first and last members of (*i*)

We get $$\frac{dx}{1} = \frac{dz}{\log(x+y)} \text{ or } dx = \frac{dz}{\log a}$$

(Using (*ii*) *i.e.*, putting value of $x + y$)

$\Rightarrow$ $$(\log a)\, dx = dz$$

Integrating $(\log a)\, x = z + b$

$\Rightarrow$ $$(\log (x + y))\, x = z + b$$

$\Rightarrow$ $$x \log (x + y) - z = b \quad ...(iii)$$

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u(x, y, z) = x + y$

$v(x, y, z) = x \log (x + y) - z$

$\therefore$ The general sol is $f(x + y, x \log (x + y) - z) = 0$

or $$x \log (x + y) - z = f(x + y)$$

where f is any arbitrary function

(*ii*) We are given the differential equation

$(p - q)(x + y) = z$

or $(x + y)\, p - (x + y)\, q = z$

Compare it with $Pp + Qq = R$

where $P = x + y$, $Q = -(x + y)$, $R = z$

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(i)$$

or $$\frac{dx}{x+y} = \frac{dy}{-(x+y)} = \frac{dz}{z}$$

From first and second members of (*i*)

we get $dx = -dy$ or $dx + dy = 0$ ...(*ii*)

Integrating, $x + y = a$

From first and last members of (*i*)

we get $\frac{dx}{x+y} = \frac{dz}{z} \Rightarrow \frac{dx}{a} = \frac{dz}{z}$ (using (

$\Rightarrow \quad dx = a\,\frac{dz}{z}$

Integrating $x = a \log|z| + b.$

$\Rightarrow \quad x = (x+y)\log|z| + b$ $(\because a = x + y$ from (i

$\Rightarrow \quad x - (x+y)\log|z| = b$ ...(i

From (*ii*) and (*iii*), we get $u = a$ and $v = b$

where $u(x, y, z) = x + y,\ v(x, y, z) = x - (x+y)\log|z|$

$\therefore$ The general sol is given by

$f(x+y, x-(x+y)\log|z|) = 0$

(*iii*) We are given the differential equation

$xzp + yzq = xy$

Compare it with $Pp + Qq = R$

Here $P = xz,\ Q = yz,\ R = xy$

$\therefore$ The auxiliary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

or $\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$ ...(i

From first two members of (*i*), we get

$\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log|x| = \log|y| + \log|c|$

$\Rightarrow \quad \log|x| = \log|yc|$

$\Rightarrow \quad |x| = |yc|$

$\Rightarrow \quad x = \pm yc$

$\Rightarrow \quad \frac{x}{y} = \pm c = a$ (say) $\Rightarrow \frac{x}{y} = a$

From last two members of (*i*), we get

$\frac{dy}{yz} = \frac{dz}{xy}$ or $\frac{dy}{z} = \frac{dz}{x} \Rightarrow \frac{dy}{z} = \frac{dz}{ay}$ $(\because$ by (*ii*))

$\Rightarrow \quad ay\,dy = z\,dz$

$\Rightarrow \quad (2y)\,a\,dy = 2z\,dz$

Integrating $\quad ay^2 = z^2 + b$

$\Rightarrow \quad \frac{x}{y}y^2 = z^2 + b$ or $xy - z^2 = b$ ...(*iii*)

From (*ii*) and (*iii*), we have $u = a$ and $v = b$